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Journal of Sound and Vibration 299 (2007) 403–408
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Short Communication

Asymptotic representations of the period for the nonlinear oscillator

A. Beléndez*, A. Hernández, T. Beléndez, C. Neipp, A. Márquez

Departamento de Física, Ingeniería de Sistemas y Teoría de la Señal, Universidad de Alicante, Apartado 99, Alicante E-03080, Spain

Received 22 February 2006; received in revised form 20 July 2006; accepted 27 July 2006
Available online 20 September 2006

Abstract

The asymptotic behaviours for small and large amplitudes, A, of the period for a nonlinear oscillator, where the square of the angular frequency depends quadratically on the velocity, are obtained. These asymptotic expressions are compared with the exact period, $T(A)$, and quite an acceptable error for a wide range of amplitudes is obtained. In addition we show that the product of the amplitude and the period, $AT(A)$, reaches 2π when the amplitude tends to infinity.

Recently Mickens has published two interesting papers [1,2] about nonlinear oscillators

$$\ddot{x} + (1 + \dot{x}^2)x = 0 \tag{1}$$

with initial conditions

$$x(0) = A \quad \dot{x}(0) = 0. \tag{2}$$

In the first paper [1], he concluded that all the solutions to Eq. (1) are periodic and its angular frequency, $\omega(A)$, as a function of the initial amplitude A, shows no singularities. This implies that the exact period, $T(A)$, of this oscillator is well-defined for all values of the amplitude. In the second [2], Mickens obtained the exact expression for the period $T(A)$ and studied some of its properties. He concluded that $T(A)$ is a monotonic decreasing function of A, and that the periods for $A = 0$ and for $A \to \infty$ are 2π and 0, respectively. He also presented a small amplitude approximation for the period whose error is quite acceptable for amplitude values in the range $0 \leqslant A \leqslant 1$. Finally, he stated that a future problem would be to obtain an asymptotic representation for the exact period for large values of A.

The main purpose of this Short Communication is to investigate the asymptotic behaviours of the period for the nonlinear oscillator given by Eq. (1). We are going to obtain asymptotic representations of this period not only for large values of amplitude A but also for small amplitudes. We are also going to compare these approximations for small and large amplitudes with the exact period obtained by means of numerical integration.

*Corresponding author. Tel.: +34 6 5903651; fax: +34 6 5903464.
E-mail address: a.belendez@ua.es (A. Beléndez).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2006.07.012

In order to obtain an asymptotic representation for small amplitudes we consider the expression for the exact period, $T(A)$, for the nonlinear oscillator given by Eq. (1) taking into account the initial conditions in Eq. (2). This expression is [2,3]

$$T(A) = 4\int_0^A \frac{\mathrm{d}x}{\sqrt{\mathrm{e}^{(A^2-x^2)}-1}}. \tag{3}$$

The linear transformation, $x = Au$, reduces this equation to the form

$$T(A) = 4A\int_0^1 \frac{\mathrm{d}u}{\sqrt{\mathrm{e}^{A^2(1-u^2)}-1}}. \tag{4}$$

For very small values of the amplitude A it is possible to take into account the following approximation:

$$\mathrm{e}^{A^2(1-u^2)} - 1 = A^2(1-u^2) + \tfrac{1}{2}A^4(1-u^2)^2 + O(A^6) \tag{5}$$

and Eq. (4) can be approximated by the expression

$$T(A) \approx T_{s1}(A) = 4A\int_0^1 \frac{\mathrm{d}u}{\sqrt{A^2(1-u^2)+\frac{1}{2}A^4(1-u^2)^2}}, \tag{6}$$

where $T_{s1}(A)$ is the asymptotic period for very small amplitudes. Eq. (6) can be rewritten as

$$T_{s1}(A) = 4\sqrt{\frac{2}{2+A^2}}\int_0^1 \frac{\mathrm{d}u}{\sqrt{(1-u^2)\left(1-\dfrac{A^2}{2+A^2}u^2\right)}}. \tag{7}$$

The integral in Eq. (7) can be written in terms of a complete elliptic integral of the first kind $K(q)$ [3,4] and then the approximate period can be obtained as

$$T_{s1}(A) = 4\sqrt{\frac{2}{2+A^2}}K(q), \tag{8}$$

where q is defined as follows:

$$q = \sqrt{\frac{A^2}{2+A^2}}. \tag{9}$$

For very small values of A it is easy to see that q will also be small and then it is possible to consider the following approximation for $K(q)$:

$$K(q) = \frac{\pi}{2}\left(1+\frac{1}{4}q^2\right). \tag{10}$$

Taking into account Eqs. (8)–(10), it follows that an asymptotic representation for the period, T_{s1}, for very small amplitudes is

$$T_{s1}(A) = \frac{\sqrt{2}(8+5A^2)}{2(2+A^2)^{3/2}}\pi. \tag{11}$$

It is easy to see that $T_{s1}(0) = 2\pi$ and if we expand in a power series Eq. (11) we obtain

$$T_{s1}(A) \approx 2\pi\left(1-\frac{A^2}{8}\right), \tag{12}$$

which is the same expression that can be obtained if we expand in a power series the approximate period obtained by means of the first-order balance method (see for example Eq. (2) in Ref. [1] or Eq. (28) in Ref. [5]).

The approximation in Eq. (6) allows us to obtain an approximate expression for the period for small values of A. However, it is difficult to get a higher order approximation from this equation. Higher order approximations for small amplitudes can be found as follows. We do the power series expansion of the integrand of Eq. (4) (by including the factor $4A$) around the point $A = 0$ and we obtain

$$\frac{4A}{\sqrt{e^{A^2(1-u^2)}-1}} = \frac{4}{\sqrt{1-u^2}}\left(1-\frac{1}{4}(1-u^2)A^2+\frac{1}{96}(1-u^2)^2A^4+\frac{1}{384}(1-u^2)^3 \times A^6 - \frac{1}{10240}(1-u^2)^4A^8+\cdots\right). \tag{13}$$

Integrating Eq. (13) between 0 and 1 a more accurate expression for the period for small values of A is obtained, T_{s2}

$$T_{s2} = 2\pi\left(1-\frac{1}{8}A^2+\frac{1}{256}A^4+\frac{5}{6144}A^6-\frac{7}{262144}A^8+\cdots\right). \tag{14}$$

Comparing this with the exact value of the period calculated numerically, it can be seen that the relative error of the approximate values for small amplitudes is less than 1% for $A<1.12$ if Eq. (11) is used, and for $A<1.92$ if Eq. (14) is used. Considering Eq. (14), the relative error for $A = 1.12$ is 0.004%, whereas taking Eq. (11), the relative error for $A = 1.92$ is 11%. It is obvious that if more terms are taken in Eq. (13) it is possible to obtain higher order approximations for small amplitudes. Thus, for example, if we take up to term A^{14} the relative error is less than 1% for $A = 2.2$.

Now we are going to obtain an asymptotic representation for large amplitudes. Firstly we consider the expression for the exact period $T(A)$ given in Eq. (3). If we make the following change of variable in Eq. (3):

$$A^2 - x^2 = z, \tag{15}$$

then

$$dx = -\frac{1}{2\sqrt{A^2-z^2}}\,dz. \tag{16}$$

Introducing Eqs. (15) and (16) into Eq. (3) gives

$$T(A) = \frac{1}{A}\int_0^{A^2}\frac{2\,dz}{\sqrt{1-\frac{z}{A^2}}\sqrt{e^z-1}}. \tag{17}$$

Now we do the power series expansion of $(1 - z/A^2)^{-1/2}$ and we obtain

$$\frac{1}{\sqrt{1-\frac{z}{A^2}}} = 1+\frac{z}{2A^2}+\frac{3z^2}{8A^4}+\frac{5z^3}{16A^6}+\cdots. \tag{18}$$

Then, the integrand of Eq. (17) can be approximated by

$$\frac{2}{\sqrt{1-\frac{z}{A^2}}\sqrt{e^z-1}} = \frac{2}{\sqrt{e^z-1}}\left(1+\frac{z}{2A^2}+\frac{3z^2}{8A^4}+\frac{5z^3}{16A^6}+\cdots\right). \tag{19}$$

This approximation needs to be integrated with respect to z from 0 to A^2. But, due to the exponential decay in the integrand, only exponentially small errors are incurred if we extend the upper limit of integration all the way to infinity. Then, we can calculate the approximate period for large amplitudes using the integral

$$\frac{1}{A}\int_0^{\infty}\frac{2\,dz}{\sqrt{1-\frac{z}{A^2}}\sqrt{e^z-1}} = \frac{1}{A}\int_0^{\infty}\frac{2}{\sqrt{e^z-1}}\left(1+\frac{z}{2A^2}+\frac{3z^2}{8A^4}+\frac{5z^3}{16A^6}+\cdots\right)dz. \tag{20}$$

For very large values of A, Eq. (20) can be approximated by the expression

$$T_{l1}(A) = \frac{1}{A}\int_0^{\infty} \frac{2\,dz}{\sqrt{e^z - 1}} = \frac{2\pi}{A}, \tag{21}$$

where $T_{l1}(A)$ is an asymptotic period for very large amplitudes. A more accurate approximation can be obtained integrating Eq. (20) and the result is

$$T_{l2}(A) = \frac{a_1}{A} + \frac{a_3}{A^3} + \frac{a_5}{A^5} + \frac{a_7}{A^7} + \cdots, \tag{22}$$

where

$$a_1 = 2\pi = 6.283185308, \tag{23}$$

$$a_3 = 2\pi \log 2 = 4.355172181, \tag{24}$$

$$a_5 = \frac{\pi}{4}\left(\pi^2 + 3(\log 4)^2\right) = 12.27973215, \tag{25}$$

$$a_7 = \frac{5\pi}{8}\left(\pi^2 \log 4 + (\log 4)^3 + 12\zeta(3)\right) = 60.41882910. \tag{26}$$

In Eq. (26), $\zeta(s)$ is the Riemann Zeta function defined as [6]

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \tag{27}$$

for $s > 1$.

From Eqs. (21) and (22) it is easy to see that

$$\lim_{A\to\infty} T(A) = 0. \tag{28}$$

Another important conclusion from Eqs. (21) and (22) is

$$\lim_{A\to\infty} (AT(A)) = 2\pi. \tag{29}$$

When this is compared to the exact value for the period calculated numerically, it can be seen that the relative error of the approximated values for large amplitudes is less than 1% for $A > 8.46$ using Eq. (21), and for $A > 2.62$ using Eq. (22). Considering Eq. (22) the relative error for $A = 8.46$ is 0.0003%, whereas taking Eq. (21), the relative error for $A = 2.62$ is 14%.

Table 1
Comparison of the asymptotic representations of the period for very small amplitudes, T_{s1} (Eq. (11)), and very large amplitudes, T_{l1} (Eq. (21)), with exact period, T (Eq. (3)), and relative errors

A	$T(A)$	$T_{s1}(A)$	$T_{l1}(A)$	Relative error (%)
0.01	6.2831	6.2831		0.0
0.1	6.2753	6.2753		0.0
1	5.5272	5.5577		0.6
1.11	5.3589	5.4136		1.0
1.28	5.0802	5.1829		2.0
1.57	4.5587	4.7856		5.0
4.09	1.6168		1.5362	5.0
5	1.2966		1.2566	3.0
6.06	1.0581		1.0368	2.0
8.42	0.7538		0.7462	1.0
10	0.6328		0.6283	0.7
100	0.062836		0.062832	0.006

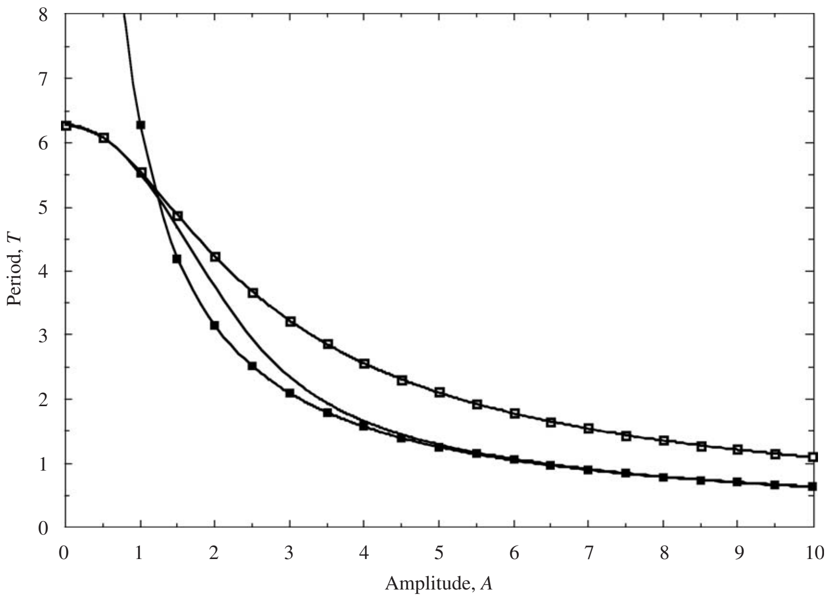

Fig. 1. Comparison of the asymptotic representations of the period for very small amplitudes, T_{s1}, —□— (Eq. (11)) and very large amplitudes, T_{l1}, —■— (Eq. (21)), with exact period, T, — (Eq. (3)).

Table 2
Comparison of the asymptotic representations of the period for small amplitudes, T_{s2} (Eq. (14)), and large amplitudes, T_{l2} (Eq. (22)), with exact period, T (Eq. (3)), and relative errors

A	$T(A)$	$T_{s2}(A)$	$T_{l2}(A)$	Relative error (%)
0.01	6.2831	6.2831		0.0
0.1	6.2753	6.2753		0.0
1	5.5272	5.5273		0.005
1.92	3.9069	3.9466		1.0
2.06	3.6543	3.7286		2.0
2.26	3.3140	3.4791		5.0
2.36	3.1553		3.3096	5.0
2.46	3.0052		3.0938	3.0
2.53	2.9054		2.9619	2.0
2.62	2.7836		2.8111	1.0
10	0.6328		0.6328	0.0
100	0.062836		0.062836	0.0

Table 1 and Fig. 1 present the comparison of asymptotic representations for very small (Eq. (11)) and very large (Eq. (21)) amplitudes with an accurate numerical integration of the period given in Eq. (3). Only for values of A between 1.11 and 8.42 is not possible to obtain the period with a relative error less than 1% using these asymptotic representations. Finally, Table 2 and Fig. 2 present the comparison of asymptotic representations for small (Eq. (14)) and large (Eq. (22)) amplitudes with an accurate numerical integration of the period given in Eq. (3). As we can see, the error is quite acceptable for wide ranges of amplitude values. Only for values of A between 1.92 and 2.62 is not possible to obtain the period with a relative error less than 1% using the asymptotic representations presented in this Short Communication (Eqs. (14) and (22)).

In summary, asymptotic representations both for small as well as for large amplitudes have been derived for the period of the nonlinear oscillator given by Eq. (1). With these asymptotic approximations it is possible to determine the period in the ranges $A \leqslant 1.92$ and $A \geqslant 2.62$ with a relative error less than 1%. We have also seen that $T(0) = 2\pi$ and $AT(A) \to 2\pi$ when $A \to \infty$.

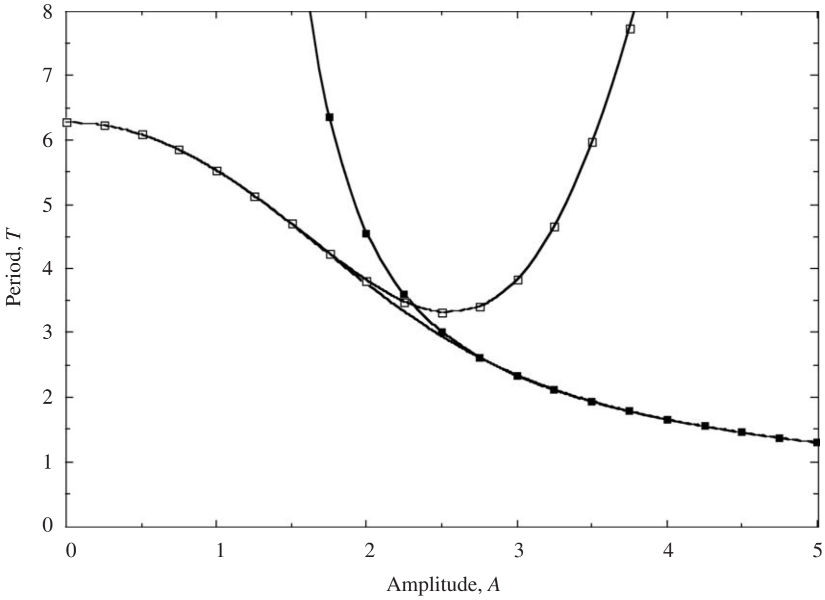

Fig. 2. Comparison of the asymptotic representations of the period for small amplitudes, T_{s2}, —□— (Eq. (14)) and large amplitudes, T_{l2}, —■— (Eq. (22)), with exact period, T, — (Eq. (3)).

Acknowledgements

This work was supported by the ''Ministerio de Educación y Ciencia'', Spain, under project FIS2005-05881-C02-02, and by the ''Generalitat Valenciana'', Spain, under project acomp006/007. We would like to thank the anonymous reviewer for his suggestions that enabled us to obtain the expressions for the asymptotic periods T_{s2} and T_{l2} presented in this paper, which have undoubtedly improved the original manuscript.

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